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# Finding passwords by random walks: how long does it take?

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## Abstract

We compare the efficiency of a deterministic ‘lawnmower’ and random search strategies for finding a prescribed sequence of letters (a password) of length  $M$  in which all letters are taken from the same  $Q$ -ary alphabet. We show that, at best, a random search takes two times longer than a ‘lawnmower’ search.

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## 1. Introduction

Suppose one has forgotten a code or a password for ones multiple-dial combination lock (or any pin-protected electronic device). Suppose next that the lock is perfect and is machined very precisely, such that when any of the discs is being rotated, it does not give any ‘click’ or any other hint when a letter or a numeral is at a correct position—this lock opens only when all the numerals or letters on all of the discs form simultaneously a correct sequence. How one should proceed in order to find a code?

An evident *brute force* approach is to explore the space of all possible combinations sequentially: starting from any random combination, one rotates one of the discs completely, step by step, from a symbol to a neighboring symbol, then turns the second disc to a neighboring symbol, rotates completely the first disc again, etc. This procedure is repeated until a correct sequence is found.

Let the desired code  $\tilde{A}$  be a sequence of  $M$  symbols:

$$\tilde{A} = \{\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \dots, \tilde{a}_M\}, \quad (1)$$

where each letter  $\tilde{a}_m$  in the sequence is taken from the same  $Q$ -ary alphabet  $\{a\}$ . With such a ‘lawnmower’ strategy, given that a rotation of any of the discs to the neighboring symbol

takes one unit of time, one is certain to find the desired code within at most  $N = Q^M$  time steps. The probability,  $P_n$ , that the code is not cracked up to the  $n$ th time step is given by

$$P_n = 1 - \frac{n+1}{N}, \quad n = 0, 1, \dots, N-1, \quad (2)$$

while the probability  $F_n$  that the code is first cracked exactly on the  $n$ th step is  $1/N$ , such that within the ‘lawnmower’ strategy the mean first-passage time  $\bar{T}_l$  to the cracking event (or the expected lifetime of the code) is simply

$$\bar{T}_l = \sum_{n=0}^{N-1} P_n = \frac{N-1}{2} \sim \frac{N}{2}. \quad (3)$$

The symbol  $\sim$  here and henceforth signifies the exact behavior to leading order in  $N$ .

In this paper, we pose a question how long will it take if, instead of a sequential exploration of all possible combinations, we search for the desired code in a random fashion. More specifically, our random search algorithm is defined as follows: we first numerate the symbols in the alphabet  $\{a\}$  and use numerals  $0, 1, 2, \dots, Q-1$  instead of symbols. Then, at each tick of the clock we choose at random a numeral along the word and add to it either  $+1$  or  $-1$ , independently on each step and with equal likelihood. At the next step, we choose again at random a numeral along the word and repeat the procedure. In original settings, it means that at each time step we choose at random a disc in our multiple-dial combination lock and rotate it downwards or upwards, with equal probability, to the neighboring symbol. Clearly, this process represents a nearest-neighbor random walk, commencing at a random site, on a periodic  $M$ -dimensional simple cubic lattice of linear size  $Q$  and comprising  $N = Q^M$  sites. The desired code  $\tilde{A}$  can be thought of as some target site on this lattice. As in the case of a ‘lawnmower’ search, we are interested to calculate the probability that the code remains not found until the  $n$ th step, the distribution of the first-passage time to the target site and the expected lifetime of the code.

## 2. Basic equations and results

Let  $a_m(n)$  denote the value of the numeral at position  $m$  along the word on the  $n$ th time step, and  $\delta(a)$  be the indicator function:

$$\delta(a) = \begin{cases} 1 & \text{for } a = 0 \\ 0 & \text{for } a \neq 0. \end{cases} \quad (4)$$

Then, the indicator function,  $I_n$ , of the event that a given trajectory of a random walk has not reached the target site  $\tilde{A}$  within the first  $n$  steps can be written as

$$I_n = \prod_{n'=0}^n \left( 1 - \prod_{m=1}^M \delta(a_m(n') - \tilde{a}_m) \right), \quad (5)$$

where

$$A(n') = \{a_1(n'), a_2(n'), a_3(n'), \dots, a_M(n')\} \quad (6)$$

denotes the random walker position on the lattice at time moment  $n'$ .

Averaging the expression in equation (5), we find that the probability that the random walk has not reached the target site up to time step  $n$  is given by

$$P_n = 1 - \frac{S_n}{N}, \quad (7)$$

where  $S_n$  is the expected number of distinct sites visited by a random walk on a periodic  $M$ -dimensional simple cubic lattice. We use here the convention that  $S_0 = 1$ . Clearly, equation (7) is an analog of equation (2), describing the form of  $P_n$  within the ‘lawnmower’ strategy. Hence, the crucial property is  $S_n$ . Explicitly, the expected number of distinct sites visited is determined as

$$S_n = \sum_{\tilde{A}} (1 - L_n(\tilde{A})), \tag{8}$$

with  $L_n(\tilde{A})$  being the probability that the simple random walk starting at the origin at time moment  $n = 0$  has not visited the site  $\tilde{A}$  up to the  $n$ th step, irrespective of the number of other sites it has visited till then. Hence,

$$L_n(\tilde{A}) = 1 - \sum_{n'=0}^n F_{n'}(\tilde{A}) \tag{9}$$

and

$$S_n = \sum_{n'=0}^n \sum_{\tilde{A}} F_{n'}(\tilde{A}), \tag{10}$$

where  $F_n(\tilde{A})$  is the probability that the first visit to the target site  $\tilde{A}$  occurred exactly on the  $n$ th step [1–3].

Using the standard results on random walks properties (see, e.g., [3, 8] and references therein), one finds eventually the following general result:

$$S_n = \frac{1}{2\pi i} \oint \frac{dz}{z^{n+1}} \frac{1}{(1-z)^2} \frac{1}{G(0; z)}, \tag{11}$$

where the integral is around the origin of the  $z$ -plane, and  $G(0; z)$  is the generating function of the probability of finding the random walk at the origin at time  $n$ , given that it started at the origin at time  $n = 0$ :

$$G(0; z) = \frac{1}{N} \sum_{\mathbf{q}} \frac{1}{1 - z\lambda(\mathbf{q})}. \tag{12}$$

In equation (12), the function  $\lambda(\mathbf{q})$  is the structure function of the random walk:

$$\lambda(\mathbf{q}) = \frac{1}{M} (\cos(q_1) + \cos(q_2) + \dots + \cos(q_M)), \tag{13}$$

while  $\mathbf{q}$  is a  $M$ -dimensional vector with components  $q_m = 2\pi k_m/Q$ , where  $k_m = 0, 1, \dots, Q - 1$  with  $Q$  being the linear size of the lattice (length of the alphabet).

In what follows, we focus on the situations when  $M > 1$  and  $Q \gg 1$ . The case  $M = 1$  corresponds to Brownian search in one-dimensional systems and has extensively been discussed recently in view of possible improvements by, e.g., intermittent random walks [4–6]. The case of binary alphabets with  $Q = 2$  describes an interesting case of search in the Hamming space and will be discussed elsewhere [7].

Consider now the form of  $P_n$  in equation (7). For sufficiently small  $n$ , each new visited site is most likely a ‘virgin’ site [3], i.e., a site visited for the first time. Hence, at short times  $S_n \sim n$  and  $P_n$  in equation (7) exhibits essentially the same behavior as its counterpart in equation (2), describing the efficiency of the ‘lawnmower’ search. Similarly, at short times, the probability  $F_n$  that the code is cracked for the first time exactly on the  $n$ th step is  $1/N$ .

At greater times, however, the growth of  $S_n$  saturates and  $S_n$  approaches  $N$ —the total number of different combinations. The relaxation of  $S_n$  to its ultimate value  $S_\infty = N$  is an

exponential function of the form

$$S_n \sim N \left( 1 - \exp \left( -\frac{n}{\tau} \right) \right), \quad \text{as } n \rightarrow \infty, \quad (14)$$

where  $\tau$  is the largest relaxation time. Calculation of  $\tau$  is a rather delicate mathematical problem, and we address the reader to [8] for more details. It was shown in [8] that for sufficiently large  $Q$ ,

$$\tau = N \begin{cases} G & \text{for } M \geq 3 \\ \ln(cN)/\pi & \text{for } M = 2, \end{cases} \quad (15)$$

where  $G$  and  $c$  are constants:  $c \approx 1.8456$ , while  $G$  is given by an  $M$ -fold integral:

$$G = \frac{1}{\pi^M} \int_0^\pi \cdots \int_0^\pi \frac{\prod_{m=1}^M dx_m}{1 - \lambda(\mathbf{x})} \quad (16)$$

with  $\lambda(\mathbf{x})$  defined by equation (3) (with the replacement  $q_m \rightarrow x_m$ ). One notes that  $G$  is just the mean number of visits to the origin by standard nearest-neighbor random walk, commencing at the origin, on a  $M$ -dimensional infinite simple cubic lattice within an infinite time.

Therefore, in the large- $n$  limit, we obtain, in virtue of equations (7) and (14), that

$$P_n \sim \exp \left( -\frac{n}{\tau} \right), \quad (17)$$

and hence, since  $F_n = P_n - P_{n+1}$ , the first-passage time distribution  $F_n$  has also an exponential tail with the characteristic decay time  $\tau$ .

The mean first-passage time  $\overline{T}_r$  to the cracking event or the lifetime of the code can be determined exactly from equations (7) and (11),  $\overline{T}_r = \sum_{n=0}^\infty P_n$ . It appears that  $\overline{T}_r$  [9] coincides with the largest relaxation time  $\tau$ , equation (15). Comparison of  $\tau$ , equation (15), and of  $\overline{T}_l$  in equation (3) allows us to draw the following conclusions:

- For this problem the ‘lawnmower’ search always outperforms a ‘random’ search algorithm.
- The worst performance of a ‘random’ search is for ‘two-letter’ codes since here the mean first-passage time  $\tau$  contains an additional logarithmic factor  $\ln(N)$  compared to the ‘lawnmower’ result.
- For three (and longer) -letter codes the mean first-passage time  $\tau$  scales linearly with  $N$ , i.e. exactly as  $\overline{T}_l$  does. However,  $\tau$  is always larger than  $\overline{T}_l$  due to a numerical factor  $f = 2G$ .  $G$  is a decreasing function of the code length; for example, for three-letter codes  $G \approx 1.516$ , for four-letter codes  $G \approx 1.239$ , for five-letter codes  $G \approx 1.156$ , etc. For larger  $M$ , the following asymptotic expansion holds [10]:

$$G = 1 + \frac{1}{2M} + \frac{3}{4M^2} + \mathcal{O} \left( \frac{1}{M^3} \right). \quad (18)$$

Hence, the ratio  $\tau/\overline{T}_l \rightarrow 2$  when the length of the code increases; it thus takes at best two times longer to crack a code using a random search than within the ‘lawnmower’ search.

Finally, we discuss a little bit different random algorithm in which, after choosing at random a numeral in the code, we increment it with equal likelihood by  $\delta = \pm 1, \pm 2, \pm 3, \dots, \pm l$ . It means that after having chosen a disc, we turn it upwards or downwards on any integer distance within an interval  $[1, l]$ . Clearly, for such an algorithm all the results in equations (7)–(12), as well as equations (14) and (15), still hold, except for the definition of  $\lambda(\mathbf{q})$ . In this more general case, the structure function of the random walk is given by

$$\lambda(\mathbf{q}) = \frac{1}{lM} \sum_{m=1}^M \sum_{j=1}^l \cos(j q_m), \quad (19)$$

while  $\tau$  is defined by equation (15) with

$$G = G_l = \frac{1}{\pi^M} \int_0^\pi \dots \int_0^\pi \prod_{m=1}^M dx_m \left( 1 - \frac{1}{lM} \sum_{m=1}^M \sum_{j=1}^l \cos(jx_m) \right)^{-1}. \quad (20)$$

Some straightforward analysis shows that  $G_l$  is a monotonically decreasing function of  $l$ . One readily finds an expansion similar to that in equation (18):

$$G_l \approx 1 + \frac{1}{2Ml}. \quad (21)$$

Hence, such a random algorithm appears to be more efficient, for large  $l$ , than the  $l = 1$  case, and  $G$  can be made very close to unity for any  $M$ . On the other hand, this algorithm cannot outperform the ‘lawnmower’ search and within the former it will take at least two times longer to find a code compared to the latter one.

### 3. Conclusions

To conclude, we have compared the efficiency of a deterministic ‘lawnmower’ and of random search strategies for finding a prescribed sequence of letters—a password—in words of length  $M$  with letters taken from the same  $Q$ -ary alphabet. We have shown that at best a search within a random strategy takes two times longer than within a ‘lawnmower’ search.

We note that the search of a password—a given sequence of letters—in the sequence space can be viewed as a (random) walk on a single-connected graph. Here, each node of the graph corresponds to a particular configuration of the lock while each bond corresponds to a physically possible one-step transformation of the lock. Clearly, for any such graph possessing a Hamiltonian cycle, the ‘lawnmower’ search for a random target site outperforms random search. The question is by how many times? Graphs considered in this paper are examples of strongly regular graphs [11], and we suppose that in a general case the answer for the question can be done in terms, for instance, of the eigenvalues of the graph.

We finally remark that the problem discussed here can be viewed from a different perspective (see [12] for more details). Suppose one has a polymer containing  $M$  monomeric units, and each of these units can be of  $Q$  different types. Starting from a particular sequence, one allows then for mutations of the monomers from one type to another. The ‘goal’ of the polymer is to attain some specific (‘foldable’ in [12]) configuration. In terms of our model, this process represents a random search algorithm in which rotation of any of the discs on an arbitrary distance is allowed, and several discs can be rotated simultaneously.

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